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Determination of a Heteroclinic Orbit in a Second Order Nonlinear Differential Equation

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Abstract: In this paper, initially, we discussed about the presence of heteroclinic orbit which is a typical kind of solution curve in a second order nonlinear differential equation by determining the explicit equation of the heteroclinic orbit. Then we added a nonlinear damping term (which contains a parameter) to the system and examined the presence of heteroclinic cycle at the zero value of the parameter by determining the stable and unstable manifolds at the saddle points.

Keywords: Local and Global bifurcation, stable and unstable manifolds, homoclinic and heteroclinic points.

1. Introduction:

The term bifurcation is commonly used in the study of nonlinear dynamics to describe any sudden qualitative change in the behavior of the system as some parameter is varied. Bifurcation refers to the splitting of the behavior of the system into two regions: one above, the other below the particular value at which the qualitative change in behavior occurs.

The theory of nonlinear systems is not yet sufficiently developed to allow us to tell in advance about what type of transition will occur for which range of parameter values for a given system. At present the transition from regularity to chaos can be categorized via two types viz local bifurcation and global bifurcation. Local bifurcation is detected through the analysis of the Jacobians associated with equilibria or cycles. But the method does not work in case of global bifurcation analysis.

Some of the theory involved in the bifurcations to chaos for flows and maps is a result of the behaviour of the stable and unstable manifolds of saddle points. The stable and unstable manifolds can form homoclinic and heteroclinic orbits as a parameter is varied. It is also possible for the stable and unstable manifolds to approach one another and eventually merge as a parameter is varied. When this occurs, there is said to be a homoclinic (or heteroclinic) intersection. The intersection is homoclinic if a stable/unstable branch of a saddle point crosses the unstable/stable branch of the same saddle point, and it is heteroclinic if the stable/unstable branches of one saddle point cross the unstable/stable branches of a different saddle point. Heteroclinic bifurcation is a global bifurcation and it is characterized by the presence of a trajectory connecting an equilibrium with another equilibrium. Interested readers can go through [2, 3, 5, 6] for thorough discussions on different kinds of bifurcations and related phenomena.

The rest of the paper is organized as follows. In section-2, we have discussed about the existence of a heteroclinic orbit in a second order nonlinear differential equation which in fact is the

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mathematical model of a nonlinear capacitor-resistor electrical circuit. Main intention of the discussion of this section is to clarify what heteroclinic orbit actually is. In the last part of this section we have spelt out the problem in which we have investigated. Section -3 contains a brief review about the existence of stable and unstable manifolds at a saddle point. In section-4, following method mentioned in [1], we have calculated the stable and unstable manifolds for our proposed problem. Finally, in section 5 we have provided our conclusion.

2. Study of the system:

A nonlinear capacitor-resistor electrical circuit can be modeled using the differential equations

$$\dot{x} = y,$$

$$\dot{y} = -x + x^3 - (a_0 + x)y$$

where a_0 is a constant and $x(t)$ represents the current in the circuit at time t. [4]

It is already known that the system,

$$\dot{x} = y, \quad \dot{y} = -x + x^3 \tag{1}$$

has a heteroclinic orbit [3, 6].

Our focus is to investigate whether this type of orbit still exist if we introduce a nonlinear damping term $-(a_0 + x)y$ to the system (1).

Before proceeding further, let us see in detail how to find the presence of heteroclinic orbit to the system (1).

The equilibrium points of the system given by (1) are found by putting $(\dot{x}, \dot{y}) = (0,0)$ which are found to be

$$P(-1,0), \quad O(0,0) \quad and \quad Q(1,0).$$

Now, to study the behavior of the system locally near the equilibrium points, we first find the Jacobian matrix which is found to be,

$$J = \begin{pmatrix} 0 & 1 \\ -1 + 3x^2 & 0 \end{pmatrix}$$

Next, we find the eigenvalues of this Jacobian Matrix at each of the above critical points to know their respective stability nature.

The eigenvalues of J at $P(-1,0)$ are found to be $\lambda_1 = -\sqrt{2}$ and $\lambda_2 = \sqrt{2}$ i.e. both real and opposite signs, so the point P is a saddle point.

The linear manifolds are determined by the eigenvectors corresponding to these eigenvalues. The eigenvector for λ_1 is $[-{}^{1}/_{\sqrt{2}}, 1]^T$ and the eigenvector for λ_2 is $[{}^{1}/_{\sqrt{2}}, 1]^T$.

Similarly we found that the equilibrium point $Q(1,0)$ is also a saddle point and the corresponding stable and unstable linear manifolds are also given by the eigenvector $[-{}^{1}/_{\sqrt{2}}, 1]^T$ and $[{}^{1}/_{\sqrt{2}}, 1]^T$ respectively.

The eigenvalues of the Jacobian matrix at the origin are found to be $\pm i$ i.e. purely imaginary. So, we conclude that we have a centre at the origin. The linear manifolds at the origin are also determined by the eigenvectors corresponding to the eigenvalues.

The phase paths satisfy the differential equation,

$$\frac{dy}{dx} = \frac{x^3 - x}{y} \qquad (2)$$

Intergrating which we get,

$$2y^2 = x^4 - 2x^2 + c, \qquad c \text{ being the constant of integration} \qquad (3)$$

A heteroclinic orbit connects the saddle point $P(-1,0)$ to the saddle point $Q(1,0)$ i.e. a heteroclinic orbit passes through $P(-1,0)$ and $Q(1,0)$.

If (3) passes through $P(-1,0)$, then $c = 1$ and eqn (3) gives,

$$y = \pm \frac{x^2 - 1}{\sqrt{2}}$$

Thus, we get two ODE's , $y = \frac{x^2-1}{\sqrt{2}}$ and $y = -\frac{x^2-1}{\sqrt{2}}$

$y = \frac{x^2-1}{\sqrt{2}}$ gives the orbit or the trajectory,

$$\frac{x+1}{x-1} = e^{-\sqrt{2}(t+t_0)} \qquad (4)$$

Thus, (4) gives a heteroclinic orbit connecting $P(-1,0)$ to $Q(1,0)$ i.e. it represents the lower heteroclinic orbit in the phase portrait (Fig 1).

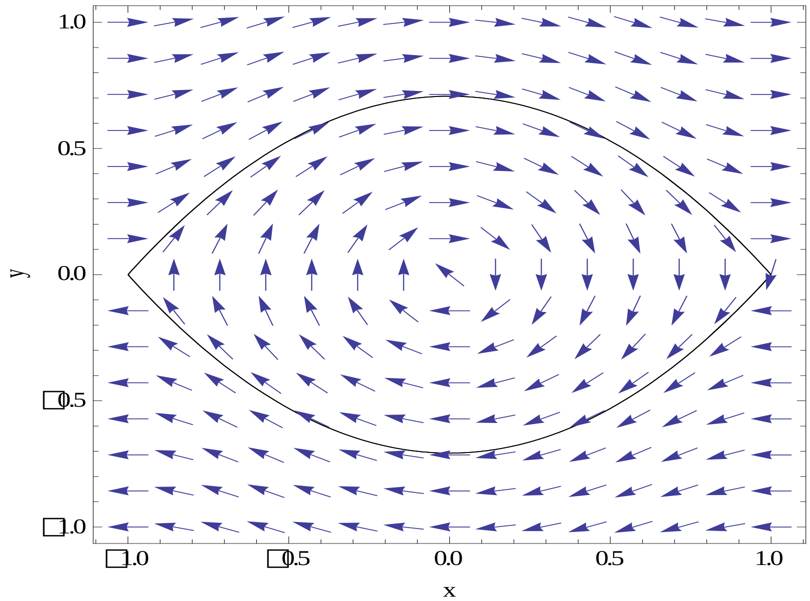

Fig 1: Phase Plot of the undamped system (1)

When $t \to \infty$, $x \to -1$ and when $t \to -\infty$, $x \to 1$.

Again $y = -\frac{x^2-1}{\sqrt{2}}$ gives the orbit or the trajectory,

$$\frac{x-1}{x+1} = e^{-\sqrt{2}(t+t_0)} \tag{5}$$

When $t \to \infty$, $x \to 1$ and when $t \to -\infty$, $x \to -1$.

Similarly, (5) gives a heteroclinic orbit connecting $Q(1,0)$ to $P(-1,0)$ i.e. it represents the upper heteroclinic orbit in the phase portrait (Fig 1).

It is to be noted that these heteroclinic orbits are typical in the sense that though they look like closed orbits, in fact they do not correspond to periodic solutions, because the trajectories takes forever or infinite time to reach a saddle point from another saddle point.

After showing explicitly the existence heteroclinic orbits in (1), next, we introduced a nonlinear damping term $(a_0 + x)y$ to the system (1) and have investigated whether heteroclinic orbits still exist and if so for what value of the parameter a_0. For this we determined the nonlinear stable and unstable manifolds at the saddle points P and Q.

3. Existence of nonlinear stable and unstable manifolds:

The linear stable and unstable manifolds E_s and E_u of a hyperbolic stationary solution, consisting of the eigenvectors of the stable and unstable eigenvalues respectively, are the tangent spaces of their nonlinear counterparts, at the origin. The existence of the nonlinear stable and unstable manifolds are ascertained by the Invariant Manifold Theorem.

Statement of the Invariant Manifold Theorem:

Consider the equation, $\dot{x} = Ax + g(x), x \in R^n$, where A has n eigenvalues λ_j, and $Re\lambda_j \neq 0$; $g(x)$ is a C^k in a neighborhood of $x = 0$ and

$$\lim_{||x||} \frac{||g(x)||}{||x||} = 0$$

Then there exists a neighborhood U of the origin, a C^k manifold W^s and a C^k function

$h^s: \pi^s(U) \to E_u$, where $\pi^s(U)$ is the projection of U into E_s, such that

1. $h^s(0) = 0$ and $\frac{\partial h^s}{\partial x_s}(0) = 0$, W^s is the graph of h^s.
2. $x(t_0) \in W^s => x(t) \in W^s \; \forall t \geq t_0$.
3. $x(t_0) \notin W^s => \exists \delta > 0, t_1 \geq t_0, \left|\left|x(t)\right|\right| > \delta \; \forall t \geq t_1$.

There also exists a C^k manifold W^u and a C^k function

$h^u: \pi^u(U) \to E_s$, where $\pi^u(U)$ is the projection of U into E_u, such that

1. $h^u(0) = 0$ and $\frac{\partial h^u}{\partial x_u}(0) = 0$, W^u is the graph of h^u.
2. $x(t_0) \in W^u => x(t) \in W^u \; \forall t \leq t_0$.
3. $x(t_0) \notin W^u => \exists \delta > 0, t_1 \leq t_0, \left|\left|x(t)\right|\right| > \delta \; \forall t \leq t_1$.

4. Computation of stable and unstable manifolds :

The next question comes, how do we compute the stable and unstable manifolds? For this, we derive an equation for h^s (and h^u) and approximate these functions by quadratic and higher order Taylor polynomials. The equations can be written as,

$$\dot{x} = Ax + f(x, y), x \in E^s$$

$$\dot{y} = By + g(x, y), y \in E^u$$

We substitute $y = h^s$, by the chain rule we get

$$D_x h^s \dot{x} = By + g(x, h^s)$$

$$\text{Or } \; h'\{Ax + f(x, h)\} = By + g(x, h),$$

where we dropped the superscript and it is to be noted that all functions are functions of x only. This is the equation satisfied by $h^s(x)$. Similarly, $h^u(y)$ satisfies the equation,

$$h'\{By + g(h, y)\} = Ah + f(h, y)$$

Stable and unstable manifolds at $P(-1,0)$:

We used the transformation of axes by shifting the origin to $P(-1,0)$. Then the system gets transformed into the form,

$$\dot{x} = y, \quad \dot{y} = -(x+1) + (x+1)^3 - (a_0 + x + 1)y$$

Here, $A = 0, f(x,y) = y$ and $B = -(a_0 + 1), g(x,y) = -(x+1) + (x+1)^3 - xy$

Let the stable manifold of P be given by the graph of the function,

$$h(x) = m_1(x+1)^2 + m_2(x+1)^3 + m_3(x+1)^4 + \dots\dots\dots$$

For a stable manifold, we have, $h'\{Ax + f(x,h)\} = Bh + g(x,h)$

$$=> \{2m_1(x+1) + 3m_2(x+1)^2 + 4m_3(x+1)^3 + \dots\}\{m_1(x+1)^2 + m_2(x+1)^3 + m_3(x+1)^4 + \dots\}$$

$$= -(x+1) + (x+1)^3 - (a_0 + x + 1)\{m_1(x+1)^2 + m_2(x+1)^3 + m_3(x+1)^4 + \dots\}$$

Comparing the coefficients of $(x+1)$ on both sides we get $0 = -1$, which is an absurdity i.e. no such function "h" exists. Thus, the stable manifold of P is the same as its linear counterpart. Similarly, the unstable manifold of P is the same as its linear counterpart which are found as follows:

The Jacobian for the damped system is

$$J = \begin{pmatrix} 0 & 1 \\ -1 + 3x^2 - y & -(a_0 + x) \end{pmatrix}$$

At the saddle point $P(-1,0)$,

$$J = \begin{pmatrix} 0 & 1 \\ 2 & -(a_0 - 1) \end{pmatrix}$$

The eigenvalues are, $\lambda_{\pm} = \frac{-(a_0-1) \pm \sqrt{(a_0-1)^2 + 8}}{2}$, where λ_+ and λ_- corresponds to the linearized unstable manifold and linearized stable manifold at $P(-1,0)$ respectively.

On putting $a_0 = 0$, we get, $\lambda_+ = 2 \; and \; \lambda_- = -1$.

The eigenvector corresponding to λ_+ is $[-1,2]^T$, so the stable manifold of P is $y = -\frac{x+1}{2}$ and the eigenvector corresponding to λ_- is $[1,2]^T$, so the unstable manifold of $P(-1,0)$ is $y = x + 1$.

Proceeding this way, we can find that the stable and unstable manifolds of $Q(1,0)$ are $y = -(x-1)$ and $y = \frac{(x-1)}{2}$ respectively.

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Solving the stable manifold at $P(-1,0)$ i.e. $y = -\frac{x+1}{2}$ and the unstable manifold at $Q(1,0)$ i.e. $y = \frac{x-1}{2}$ together we get a orbit given by $\frac{x+1}{x-1} = e^{-t}$. This represents a heteroclinic orbit of the system as it can be verified that that this orbit satisfies the system of our study and also $t \to \infty$ gives $x \to -1$ and $t \to -\infty$ gives $x \to 1$.

Again, solving the unstable manifold at $P(-1,0)$ i.e. $y = x + 1$ and the unstable manifold at $Q(1,0)$ i.e. $y = -(x-1)$ together we get a orbit given by $\frac{x+1}{x-1} = e^{2t}$. This again represents a heteroclinic orbit of the system as it can also be verified that that this orbit satisfies the system of our study and also $t \to \infty$ gives $x \to 1$ and $t \to -\infty$ gives $x \to -1$.

Hence there is a heteroclinic cycle between the saddle points P and Q when $a_0 = 0$.

5. Conclusion :

If $a_0 > 0$, the origin is stable and P and Q are saddles. The phase plot for $a_0 = 0.02 > 0$ in fig. 2 shows the orbit spiraling inward towards the origin i.e. the origin behaves like an attractor. On the contrary, if $a_0 < 0$, the origin is seen to be unstable and P and Q are still saddles. The phase plot for $a_0 = -0.02 < 0$ is shown in fig. 3 which shows that the orbit is spiraling outward away from the origin. For the model at our hand, physically it means that if $a_0 > 0$, the current in the circuit eventually dies away to zero with increasing time, but for $a_0 < 0$, the current increases indefinitely, which is not expected physically.

But for $a_0 = 0$, a limit cycle appears as shown in the phase plot as shown in fig. 4. Since the limit cycle appears from a heteroclinic loop, so we conclude that a heteroclinic bifurcation takes place for $a_0 = 0$.

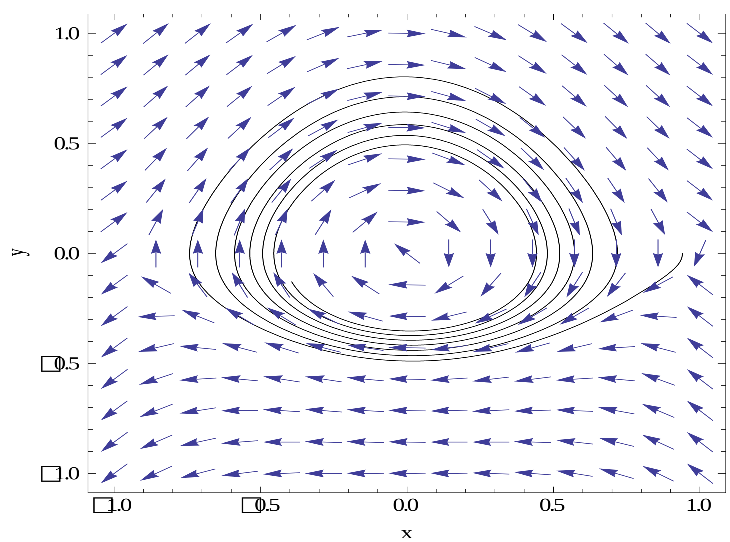

Fig 2: Phase Plot of the damped system for $a_0 = 0.02$

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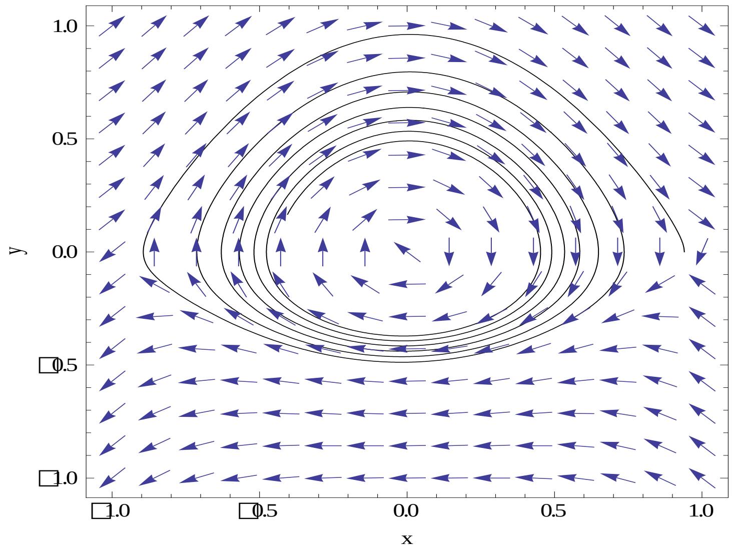

Fig 3: Phase Plot of the damped system for $a_0 = -0.02$

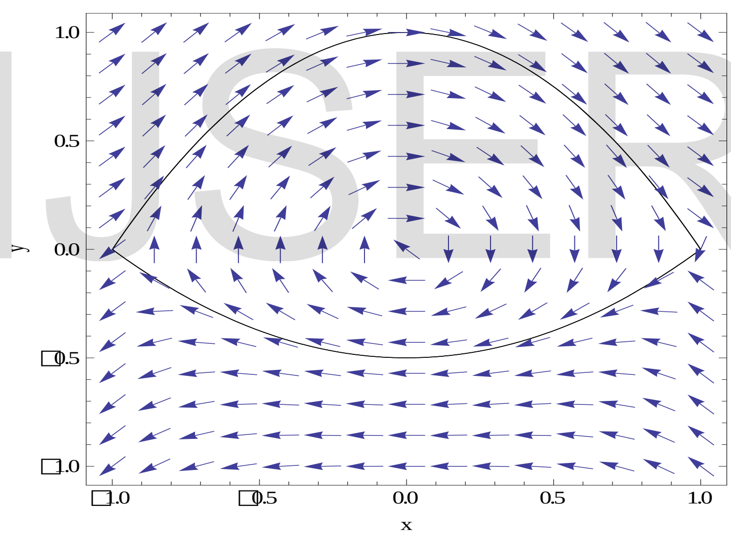

Fig 4: Phase Plot of the damped system for $a_0 = 0$

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